EXTREMAL FIRST DIRICHLET EIGENVALUE OF DOUBLY CONNECTED PLANE DOMAINS AND DIHEDRAL SYMMETRY

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Abstract. We deal with the following eigenvalue optimization problem: Given a bounded domain $D \subset \mathbb{R}^2$, how to place an obstacle B of fixed shape within D so as to maximize or minimize the fundamental eigenvalue λ_1 of the Dirichlet Laplacian on $D \setminus B$. This means that we want to extremize the function $\rho \mapsto \lambda_1(D \setminus \rho(B))$, where ρ runs over the set of rigid motions such that $\rho(B) \subset D$. We answer this problem in the case where both D and B are invariant under the action of a dihedral group $\mathbb{D}_n$, $n \geq 2$, and where the distance from the origin to the boundary is monotonous as a function of the argument between two axes of symmetry. The extremal configurations correspond to the cases where the axes of symmetry of B coincide with those of D.

1. Introduction and Statement of the main Result

The relations between the shape of a domain and the eigenvalues of its Dirichlet or Neumann Laplacian, have been intensively investigated since the 1920's when Faber [5] and Krahn [12] have proved independently the famous eigenvalue isoperimetric inequality first conjectured by Rayleigh (1877): the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of any bounded domain $\Omega \subset \mathbb{R}^n$ satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where Ω^* is a ball having the same volume as Ω. We refer to the review papers of Ashbaugh [1, 2] and Henrot [9] for a survey of recent results on optimization problems involving eigenvalues.

The present work deals with the following eigenvalue optimization problem: Given a bounded domain D, we want to place an obstacle (or a hole) B, of fixed shape, inside D so as to maximize or minimize the fundamental eigenvalue λ_1 of the Laplacian or Schrödinger operator on $D \setminus B$ with Zero Dirichlet conditions on the boundary.

In other words, the problem is to optimize the principal eigenvalue function $\rho \mapsto \lambda_1(D \setminus \rho(B))$, where ρ runs over the set of rigid motions such that $\rho(B) \subset D$.

The first result obtained in this direction concerned the case where both D and B are disks of given radii. Indeed, it follows from Hersch's work [10] that the maximum of λ_1 is achieved when the disks are concentric (see also [14]). This result has been extended to any dimension by several authors (Harrell, Kröger and Kurata [8], Kesavan [11], ...). Actually, Harrell, Kröger and Kurata [8] gave a more general result showing that, if the domain D satisfies an interior symmetry property with respect to a hyperplane P passing through the center of the spherical obstacle B

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(which means that the image by the reflection with respect to P of one component of $D \setminus P$ is contained in D), then the Dirichlet fundamental eigenvalue $\lambda_1(D \setminus B)$ decreases when the center of B moves perpendicularly to P in the direction of the boundary of D. In the particular case where both the domain D and the obstacle B are balls, this implies that the minimum of $\lambda_1(D \setminus B)$ corresponds to the limit case where B touches the boundary of D.

Notice that when the obstacle B is a disk, only translations of B may affect the λ_1 of $D \setminus B$ and the optimal placement problem reduces to the choice of the center of B inside D.

In the present work we investigate a kind of dual problem in the sense that we consider a *nonspherical* obstacle B whose center of mass is fixed inside D, and seek the optimal positions while turning B around its center.

It is of course hopeless to expect a universal solution to this problem. In fact, we will restrict our investigation to a class of domains satisfying a dihedral symmetry and a monotonicity conditions.

Thus, let D be a simply-connected plane domain and assume that the following conditions are satisfied:

(i) ($\mathbb{D}_n$-symmetry) for an integer $n \geq 2$, D is invariant under the action of the dihedral group $\mathbb{D}_n$ of order $2n$ generated by the rotation $\rho_{\frac{2\pi}{n}}$ of angle $\frac{2\pi}{n}$ and a reflection S. Such a domain admits n axes of symmetry passing through the origin and such that the angle between 2 consecutive axes is $\frac{\pi}{n}$.

(ii) (monotonicity of the boundary) the distance $d(O, x)$ from the origin to a point x of the boundary of D is monotonous as a function of the argument of x, in a sector delimited by two consecutive symmetry axes.

Notice that assumption (i) guarantees that the center of mass of D is at the origin. Regular n-gones centered at the origin are the simplest examples of domains satisfying these assumptions. More generally, if g is any positive even $\frac{2\pi}{n}$-periodic continuous function that is monotonous on the interval $(0, \frac{\pi}{n})$, then the domain

$$D = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\},$$

satisfies assumptions (i) and (ii). Actually, up to a rigid motion, any domain satisfying assumptions (i) and (ii) can be parametrized in such a manner.

It is worth noticing that, due to the monotonicity condition, the "distance to the origin" function on the boundary of D achieves its maximum and its minimum alternatively at the intersection points of ∂D with the $2n$ half-axes of symmetry. The n points of ∂D at maximal (resp. minimal) distance from the origin will be called "outer vertices" (resp. "inner vertices") of D.

Our main result is the following

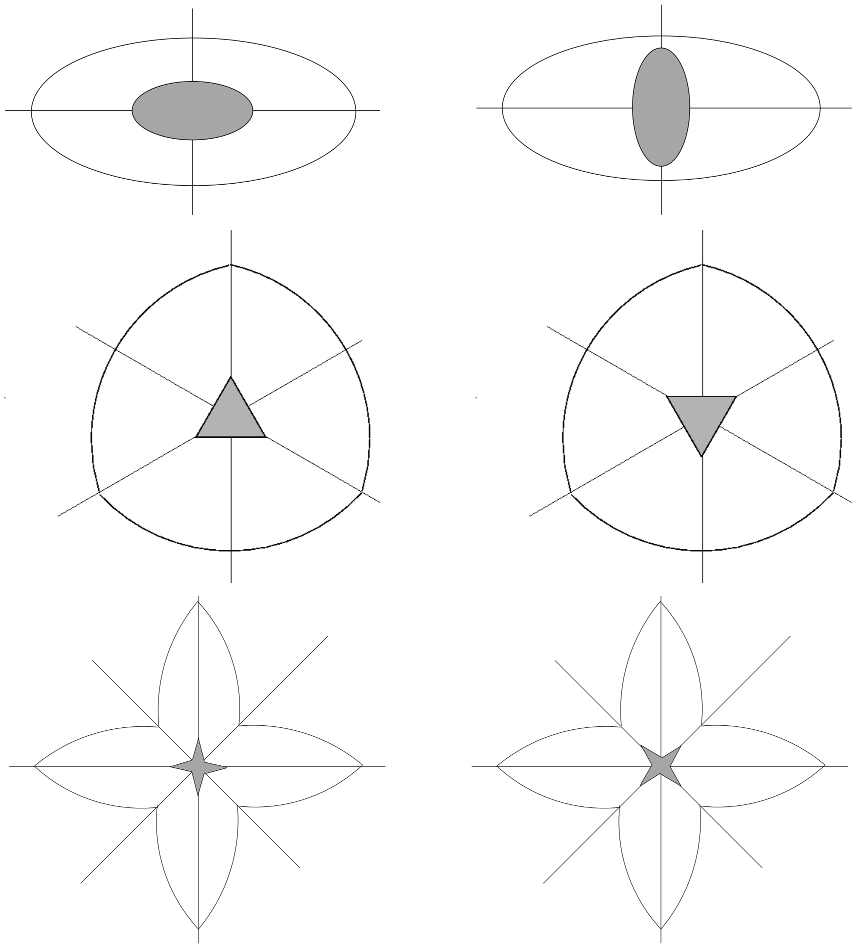
Theorem 1. *Let D and B be two plane domains satisfying the assumptions of $\mathbb{D}_n$-symmetry and monotonicity (i) and (ii) above for an integer $n \geq 2$. Assume furthermore that B has C^2 boundary and that $\rho(B) \subset D$ for all $\rho \in SO(2)$. Then, the fundamental Dirichlet eigenvalue $\lambda_1(D \setminus B)$ of $D \setminus B$ is optimized exactly when the axes of symmetry of B coincide with those of D.*

The maximizing configuration corresponds to the case where the outer vertices of B and D lie on the same half-axes of symmetry (we will then say that B occupies the "ON" position in D).

The minimizing configuration corresponds to the case where the outer vertices of B lie on the half-axes of symmetry passing through the inner vertices of D (this is what will be called the "OFF" position).

Actually, we will prove that, except for the trivial case where D or B is a disk, the fundamental Dirichlet eigenvalue of $D \setminus B$ decreases gradually when B switches from "ON" to "OFF".

The main ingredients of the proof of Theorem 1 are Hadamard's variation formula for λ_1 and the technique of domain reflection initiated by Serrin [17] in PDE's setting.



Examples of maximal (left) and minimal (right) configurations
with $n = 2$, 3 and 4 respectively

Extensions of Theorem 1 to the following situations can be obtained up to slight changes in the proof (indeed, only the Hadamard formula should be replaced by the variation formula corresponding to the new functional):

(1) Soft obstacles: instead considering the Dirichlet Laplacian on $D \setminus B$, we consider the Schrödinger type operator

$$H(\alpha, B) := \Delta - \alpha\chi_B$$

acting on $H_0^1(D)$, where $\alpha > 0$ and χ_B is the indicator function of B. Optimization problems related to the fundamental eigenvalue of operators of this kind have been investigated in particular in [8] and [3]. Under the assumptions of Theorem 1 on D and B, $\forall\alpha > 0$, the fundamental eigenvalue of $H(\alpha, B)$ achieves its maximum at the "ON" position and its minimum at the "OFF" position.

(2) Wells: this case corresponds to the operator $H(\alpha, B)$ with $\alpha < 0$. Under the circumstances of Theorem 1, $\forall\alpha < 0$, the first eigenvalue of $H(\alpha, B)$ achieves its maximum at the "OFF" position and its minimum at the "ON" position.

(3) Stationary problem : the problem now is to optimize the Dirichlet energy $J(D \setminus B) := \int_{D\setminus B} |\nabla u|^2 dx$ of the unique solution u of the problem

$$\begin{cases} \Delta u &= -1 \quad \text{in } D \setminus B \\ u &= 0 \quad\ \ \text{on } \partial(D \setminus B), \end{cases}$$

This problem was treated in [11, Section 2] in the case where both D and B are balls. Under the assumptions of Theorem 1 on D and B, one can prove that $J(D \setminus B)$ achieves its maximum when B is at the "ON" position and its minimum when B is at the "OFF" position.

2. Proof of the main result

Without loss of generality, we may assume that the domain D and the obstacle B are centered at the origin and are both symmetric with respect to the x_1-axis so that they can be parametrized in polar coordinates by

$$D = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \le r < g(\theta)\},$$

$$B = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \le r < f(\theta)\},$$

where f and g are two positive even $\frac{2\pi}{n}$-periodic functions which are *nondecreasing* on $(0, \frac{\pi}{n})$. To avoid technicalities, we suppose throughout that g is continuous and f is C^2. Extensions of our result to a wider class of domains would certainly be possible up to some additional technical difficulties.

The condition that the obstacle B can freely rotate around his center inside D, that is $\rho(\bar{B}) \subset D$ for all $\rho \in SO(2)$, amounts to the following:

$$f(\frac{\pi}{n}) = \max_{0 \le \theta \le 2\pi} f(\theta) < \min_{0 \le \theta \le 2\pi} g(\theta) = g(0).$$

Let us denote, for all $t \in \mathbb{R}$, by ρ_t the rotation of angle t, that is, $\forall\zeta \in \mathbb{R}^2 \cong \mathbb{C}$, $\rho_t(\zeta) = e^{\mathbf{i}t}\zeta$, and set

$$B_t := \rho_t(B) \text{ and } \Omega(t) := D \setminus B_t.$$

Let $\lambda(t)$ be the fundamental eigenvalue of the Dirichlet Laplacian on $\Omega(t)$. It is well known that, since it is simple, the first Dirichlet eigenvalue $\lambda(t)$ is a differentiable function of t (see [6, 15]). We denote by $u(t)$ the one parameter family of nonnegative first eigenfunctions satisfying, $\forall t \in \mathbb{R}$,

$$\begin{cases} \Delta u(t) &= -\lambda(t)u(t) \quad \text{in } \Omega(t) \\ u(t) &= 0 \quad\quad\quad\quad\ \text{on } \partial\Omega(t) \\ \int_{\Omega(t)} u^2(t) &= 1. \end{cases}$$

The derivative of $\lambda(t)$ is then given by the following so-called Hadamard formula (see [4, 6, 7, 16]):

$$\lambda'(t) = \int_{\partial B_t} \left|\frac{\partial u(t)}{\partial \eta_t}\right|^2 \eta_t \cdot v \, d\sigma, \tag{1}$$

where η_t is the inward unit normal vector field of $\partial\Omega(t)$ (hence, along ∂B_t the vector η_t is outward with respect to B_t) and v denotes the restriction to $\partial\Omega(t) = \partial D \cup \partial B_t$ of the deformation vector field. In our case, the vector v vanishes on ∂D and is given by $v(\zeta) = \mathbf{i}\zeta$ for all $\zeta \in \partial B_t$.

Since both Ω and B are invariant by the dihedral group $\mathbb{D}_n$, it follows that, $\forall t \in \mathbb{R}$, $\Omega(t + \frac{2\pi}{n}) = \Omega_t$. Moreover, if we denote by S_0 the reflection with respect to the x_1-axis, then we clearly have $\rho_{-t} = S_0 \circ \rho_t \circ S_0$ which gives $B_{-t} = S_0(B_t)$ and $\Omega_{-t} = S_0(\Omega_t)$. Hence, as a function of t, the first Dirichlet eigenvalue of Ω_t is even and periodic of period $\frac{2\pi}{n}$, that is, $\forall t \in \mathbb{R}$,

$$\lambda(t + \tfrac{2\pi}{n}) = \lambda(t) \text{ and } \lambda(-t) = \lambda(t).$$

Therefore, it suffices to investigate the variations of $\lambda(t)$ on the interval $\left[0, \frac{\pi}{n}\right]$ and Theorem 1 is a consequence of the following:

Theorem 2. *Assume that neither D nor B is a disk.*

(i) $\forall t \in \left(0, \frac{\pi}{n}\right)$, $\lambda'(t) < 0$. *Hence, $\lambda(t)$ is strictly decreasing on $\left(0, \frac{\pi}{n}\right)$.*
(ii) $\forall k \in \mathbb{Z}$, $\lambda'(k\frac{\pi}{n}) = 0$ *and* $k\frac{\pi}{n}$, $k \in \mathbb{Z}$, *are the only critical points of λ on $\mathbb{R}$.*

Hence, $\lambda(t)$ achieves its maximum for $t = 0 \mod \frac{2\pi}{n}$ which corresponds to the "ON" position, and its minimum for $t = \frac{\pi}{n} \mod \frac{2\pi}{n}$ which corresponds to the "OFF" position. Of course, if D or B is a disk, then the function $\lambda(t)$ is constant.

In what follows we will denote, for any $\alpha \in \mathbb{R}$, by z_α the $\theta = \alpha$ axis, that is $z_\alpha := \{re^{\mathbf{i}\alpha};\ r \in \mathbb{R}\}$, and by z_α^+ the half-axis $\{re^{\mathbf{i}\alpha};\ r \geq 0\}$.

We start the proof with the following elementary lemma.

Lemma 1. *Let K be a plane domain defined in polar coordinates by $K = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \leq r < h(\theta)\}$, where h is a positive 2π-periodic function of classe C^1, and let v be a vector field whose restriction to ∂K is given by*

$$v(\theta) := v(h(\theta)e^{\mathbf{i}\theta}) = \mathbf{i}h(\theta)e^{\mathbf{i}\theta} = h(\theta)e^{\mathbf{i}(\theta + \frac{\pi}{2})}.$$

We denote by η the unit outward normal vector field of ∂K. One has, at any point $h(\theta)e^{\mathbf{i}\theta}$ of ∂K where η is defined,

(i) $\eta(\theta) := \eta(h(\theta)e^{\mathbf{i}\theta}) = \frac{h(\theta)e^{\mathbf{i}\theta} - \mathbf{i}h'(\theta)e^{\mathbf{i}\theta}}{\sqrt{h^2(\theta) + h'^2(\theta)}}$
(ii) $\eta \cdot v(\theta) = \frac{-h(\theta)h'(\theta)}{\sqrt{h^2(\theta) + h'^2(\theta)}}$. *Hence, $\eta.v(\theta)$ has constant sign on an interval I if and only if h is monotonous in I.*
(iii) *if for some $\alpha > 0$, the domain K is symmetric with respect to the axis z_α, then the function $\eta \cdot v$ is antisymmetric w.r.t this axis, that is*

$$\eta \cdot v(\alpha + \theta) = -\eta \cdot v(\alpha - \theta).$$

Proof. Assertions (i) and (ii) are direct consequences from the definition of K. The fact that K is symmetric with respect to the axis z_α implies that the function h satisfies $h(\alpha + \theta) = h(\alpha - \theta)$. Therefore, (iii) follows immediately from (ii). ☐

We will denote by S_α the symmetry with respect to the axis z_α. We will also denote, for $\alpha < \beta$, by $\sigma(\alpha, \beta)$ the sector delimited by z_α^+ and z_β^+, that is

$$\sigma(\alpha, \beta) = \{re^{\mathbf{i}\theta}; r > 0 \text{ and } \alpha < \theta < \beta\}.$$

Lemma 2. *Let D be as above. For all $t \in \left(0, \frac{\pi}{n}\right)$, we have:*

$$S_{\frac{\pi}{n}+t}\left(D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2\pi}{n}+t\right)\right) \subseteq D \cap \sigma\left(t, \frac{\pi}{n}+t\right).$$

Moreover, if D is not a disk, then

$$S_{\frac{\pi}{n}+t}\left(\partial D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2\pi}{n}+t\right)\right) \cap D \neq \emptyset.$$

Proof. The action of the symmetry $S_{\frac{\pi}{n}+t}$ is given in polar coordinates by $S_{\frac{\pi}{n}+t}(re^{\mathbf{i}\theta}) = re^{\mathbf{i}(2(\frac{\pi}{n}+t)-\theta)}$. Hence,

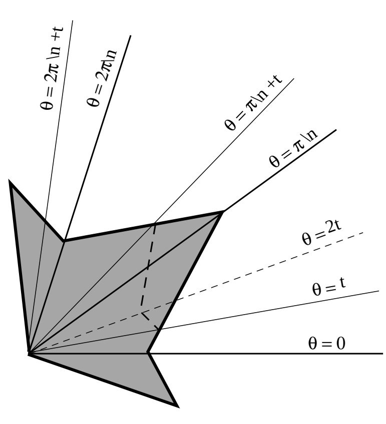
$$S_{\frac{\pi}{n}+t}\left(D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2\pi}{n}+t\right)\right) = S_{\frac{\pi}{n}+t}(D) \cap \sigma\left(t, \frac{\pi}{n}+t\right).$$

Moreover, the domain D being parametrized by a positive even $\frac{2\pi}{n}$-periodic function $g(\theta)$, that is $D = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\}$, its image $S_{\frac{\pi}{n}+t}(D)$ can be parametrized in the same manner by the function $g^*(\theta) = g(\theta - 2t)$. Thus

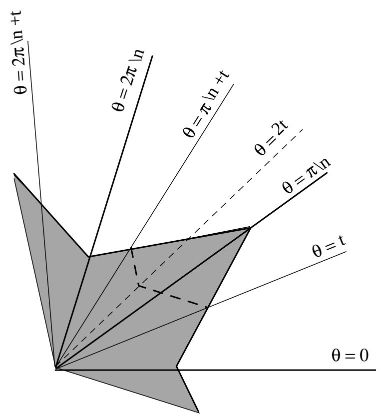
$$S_{\frac{\pi}{n}+t}(D) \cap \sigma\left(t, \frac{\pi}{n}+t\right) = \{re^{\mathbf{i}\theta}; \theta \in \left(t, \frac{\pi}{n}+t\right), 0 \leq r < g(\theta - 2t)\}.$$

Therefore, we need to prove that $F(\theta) = g(\theta) - g^*(\theta)$ is nonnegative for every θ in the interval $(t, \frac{\pi}{n}+t)$. This will be possible thanks to the assumptions of symmetry (that is g is even and $\frac{2\pi}{n}$-periodic) and monotonicity (that is g is nondecreasing on $[0, \frac{\pi}{n}]$). Indeed, these properties imply that on the interval $\left(t, \frac{\pi}{n}+t\right)$,

- g achieves its maximum at $\theta = \frac{\pi}{n}$,
- g^* achieves its minimum at $\theta = 2t$.



case $2t < \frac{\pi}{n}$



case $2t > \frac{\pi}{n}$

Four cases must be considered separately:

- If $t < \theta \leq \min\{2t, \frac{\pi}{n}\}$, we may write, since g is even, $F(\theta) = g(\theta) - g(2t - \theta)$, with $0 \leq 2t - \theta < \theta \leq \frac{\pi}{n}$. Since g is nondecreasing on $[0, \frac{\pi}{n}]$, we get $F(\theta) \geq 0$.
- If $\max\{2t, \frac{\pi}{n}\} \leq \theta < \frac{\pi}{n} + t$, we may write, since g is even and $\frac{2\pi}{n}$-periodic, $F(\theta) = g(2\frac{\pi}{n} - \theta) - g(\theta - 2t)$ with $0 \leq \theta - 2t < 2\frac{\pi}{n} - \theta \leq \frac{\pi}{n}$. Hence, $F(\theta) \geq 0$.

- If $2t < \frac{\pi}{n}$ and $2t \le \theta \le \frac{\pi}{n}$, then $0 \le \theta - 2t < \theta \le \frac{\pi}{n}$ and, then, $F(\theta) = g(\theta) - g(\theta - 2t) \ge 0$.
- If $2t > \frac{\pi}{n}$ and $\frac{\pi}{n} \le \theta \le 2t$, then $0 \le 2t - \theta < 2\frac{\pi}{n} - \theta \le \frac{\pi}{n}$ and, then, $F(\theta) = g(2\frac{\pi}{n} - \theta) - g(2t - \theta) \ge 0$.

Hence, $F(\theta)$ is nonnegative for all θ in $(t, \frac{\pi}{n} + t)$.

Now, if D is not a disk, then g is nonconstant on $[0, \frac{\pi}{n}]$. Following the arguments above, we deduce that the function $F(\theta)$ is positive somewhere on $(t, \frac{\pi}{n} + t)$ which means that $S_{\frac{\pi}{n}+t}\left(\partial D \cap \sigma\left(\frac{\pi}{n} + t, \frac{2\pi}{n} + t\right)\right)$ meets the interior of D. □

Proof of Theorem 2. Notice first that, since λ is an even and $\frac{2\pi}{n}$-periodic function of t, one immediately gets, $\forall k \in \mathbb{Z}$, $\lambda(k\frac{\pi}{n} - t) = \lambda(k\frac{\pi}{n} + t)$ and, then,

$$\lambda'\left(k\frac{\pi}{n}\right) = 0.$$

Alternatively, one can deduce that $\lambda'\left(k\frac{\pi}{n}\right) = 0$ from Hadamard's variation formula (1) after noticing that the domain $\Omega(k\frac{\pi}{n})$ is symmetric with respect to the x_1-axis and that the first Dirichlet eigenfunction $u(k\frac{\pi}{n})$ satisfies $u \circ S_0 = u$, where S_0 is the symmetry with respect to the x_1-axis.

Let us fix a t in $\left(0, \frac{\pi}{n}\right)$ and denote by u the nonnegative first Dirichlet eigenfunction of $\Omega(t)$ satisfying $\int_{\Omega(t)} u^2 = 1$. The domain $\Omega(t)$ is clearly invariant by the rotation $\rho_{\frac{2\pi}{n}}$ of angle $\frac{2\pi}{n}$, hence $u \circ \rho_{\frac{2\pi}{n}} = u$. On the other hand, the domain B being parametrized by a positive even $\frac{2\pi}{n}$-periodic function $f(\theta)$, that is $B = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \le r < f(\theta)\}$, one has

$$B_t = \{re^{\mathbf{i}\theta}; \theta \in [0, 2\pi), 0 \le r < h(\theta)\},$$

with $h(\theta) = f(\theta - t)$. Hence, the function $\eta_t \cdot v$ is invariant by $\rho_{\frac{2\pi}{n}}$ (Lemma 1) and we have (Hadamard formula (1))

$$\lambda'(t) = \int_{\partial B_t} \left|\frac{\partial u}{\partial \eta_t}\right|^2 \eta_t \cdot v \; d\sigma = n \int_{\partial B_t \cap \sigma(t, \frac{2\pi}{n} + t)} \left|\frac{\partial u}{\partial \eta_t}\right|^2 \eta_t \cdot v \; d\sigma.$$

Since B_t is symmetric with respect to the axis $z_{\frac{\pi}{n}+t}$, we have (Lemma 1), $\eta_t \cdot v(\frac{\pi}{n} + t + \theta) = -\eta_t \cdot v(\frac{\pi}{n} + t - \theta)$ or, equivalently, $\eta_t \cdot v(x) = -\eta_t \cdot v(x^*)$, where x^* denotes the symmetric of x with respect to $z_{\frac{\pi}{n}+t}$. This yields

$$\lambda'(t) = n \int_{\partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)} \left(\left|\frac{\partial u}{\partial \eta_t}(x)\right|^2 - \left|\frac{\partial u}{\partial \eta_t}(x^*)\right|^2 \right) \eta_t \cdot v(x) \; d\sigma$$

Notice that the function $h(\theta)$ is decreasing between $\frac{\pi}{n} + t$ and $\frac{2\pi}{n} + t$ and, then, $\eta_t \cdot v$ is nonnegative on $\partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)$ (Lemma 1).

Let $H(t) := \Omega(t) \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)$. Applying Lemma 2, and since B_t is symmetric with respect to the axis $z_{\frac{\pi}{n}+t}$, one gets

$$S_{\frac{\pi}{n}+t}(H(t)) \subset \Omega(t) \cap \sigma(t, \frac{\pi}{n} + t).$$

Hence, the function $w(x) = u(x) - u(x^*)$ is well defined on $H(t)$ and satisfies $w(x) = 0$ for all x in $\partial H(t) \cap \left(\partial B_t \cup z_{\frac{\pi}{n}+t} \cup z_{\frac{2\pi}{n}+t}\right)$. Moreover, since u vanishes on ∂D and is positive inside $\Omega(t)$, $w(x) \le 0$ for all x in $\partial H(t) \cap \partial D$ and $w(x) < 0$ for certain x in $\partial H(t) \cap \partial D$ (recall that D is not a disk and apply the second part of Lemma 2).

Therefore, the nonconstant function w satisfies the following:

$$\begin{cases} \Delta w &= -\lambda(t)w \quad \text{in } H(t) \\ w &\leq 0 \quad\quad\quad \text{on } \partial H(t). \end{cases}$$

Hence, w must be nonpositive on the whole of $H(t)$. Otherwise, a nodal domain $V \subset H(t)$ of w would have the same first Dirichlet eigenvalue as $\Omega(t)$. But, due to the invariance of $\Omega(t)$ by $\rho_{\frac{2\pi}{n}}$, the domain $\Omega(t)$ would contain n copies of V leading to a strong contradiction with the domain monotonicity theorem for eigenvalues. Therefore, $\Delta w \geq 0$ in $H(t)$ and w achieves its maximal value (i.e. zero) on $\partial B_t \cap \sigma(\frac{\pi}{n}+t, \frac{2\pi}{n}+t) \subset \partial H(t)$. The Hopf maximum principle (see [13, Theorem 7, ch.2]) then implies that, at any regular point x of $\partial B_t \cap \sigma(\frac{\pi}{n}+t, \frac{2\pi}{n}+t)$, one has

$$\frac{\partial w}{\partial \eta_t}(x) = \frac{\partial u}{\partial \eta_t}(x) - \frac{\partial u}{\partial \eta_t}(x^*) < 0.$$

It follows that $\lambda'(t) \leq 0$ and that the equality holds if and only if $\eta_t \cdot v \equiv 0$. By Lemma 1, this last equality occurs if and only if f is constant which means that B is a disk.

□

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